.

ALTERNATIVE PROOF OF FRULLANI'S THEOREM AND APPLICATIONS IN EVALUATING FRULLANI INTEGRALS

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Abstract: This article provides an alternative proof for the Frullani integral formula using an approach different from the existing one. This alternative proof gave us a novel method for evaluating certain improper integrals of Frullani type. Moreover, the alternative proof also obtained an exciting result relating the Frullani integral to a specific class of improper double integrals—the alternative proof started by stating and proving lemmas used as stepping stones to obtain the main proof. An essential condition was also imposed to obtain the desired result.

Keywords: Double integral, Frullani integral, improper integral, Laplace transform

1. INTRODUCTION

The main type of integral under consideration of this paper was first conceptualized by an Italian Mathematician Giuliano Frullani (1795–1834) from whence the name "Frullani integral". This integral is expressed in the form:

$$\int_0^\infty \frac{f(ax)-f(bx)}{x}\,dx = \ln\left(\frac{a}{b}\right)[f(\infty)-f(0)], \qquad (1.1)$$

where $a, b > 0$, f is a continuously differentiable function on $[0,\infty)$, $f(\infty) = \lim_{x\to\infty} f(x)$ and $f(0) = \lim_{x\to 0^+} f(x)$ (Bravo *et al.*, 2017).

In 1828, Frullani published this result but apparently with an inadequate proof. Augustin-Louis Cauchy gave a satisfactory proof under certain conditions of $f(x)$. Also, according to Ostrowski (1949), Cauchy's result has been fully generalized, replacing the limits $f(0)$ and $f(\infty)$ by suitable mean values. The proof of Cauchy is used in most textbooks today. Arias-De-Reyna (1990) gives the fullest account of the history of the discovery of the solution of this integral, which was first given by Frullani in 1821, and later by Cauchy in 1823 and 1827. Iyengar (1940) gives the first modern analysis, followed by Ostrowski (1949), Agnew (1942, 1951), Ostrowski (1976) and Arias-De-Reyna (1990).

More recently, the Frullani integral appeared in the proof for a theorem by Allouche (2007), which claims that the value of *Ramanujan integral I* can be deduced from *Ramanujan integral J*, where

$$I = \int_0^1 \frac{x^{p-1}}{1-x} - \frac{rx^{q-1}}{1-x^r}\,dx$$

and

$$J = \int_0^{\infty} \frac{(1+ax)^{-p} - (1+bx)^{-p}}{x} dx.$$

In another paper (Trainin, 2010), a generalized form of Frullani's theorem was applied in evaluating improper integrals of the form

$$I(n.m) = \int_0^{\infty} \frac{sin^n x}{x^m} dx.$$

In this paper, we shall present and prove that if a function f is of exponential order and $\int_0^{\infty} f'(x)\, dx$ is convergent then Eq. (1.1) holds. An unexpected by-product is that the process of proving the formula provided us alternative methods for evaluating Frullani integrals and certain class of improper double integrals.

2. METHODOLOGY

The method adopted in this paper was mainly expository. Definitions and theorems used can be found in Ferrar (1958), Taylor & Mann (1955), and Wrede & Spiegel (2010). Lemmas were proven first before proving the main theorem. The techniques that were used to prove the lemmas involve transformations, triangle inequality, and theorems on convergence. Proving the main theorem requires the use of theorems on convergence, integration by parts technique, and the fundamental theorem of calculus.

3. RESULTS AND DISCUSSSION

3.1 Definitions and introductory lemmas

The succeeding discussion on some definitions and results was lifted from Wrede and Spiegel (2010).

The function f is said to be of *exponential order* γ if there exist constants $\gamma \epsilon\ \mathbb{R}, M > 0, t_0 > 0$ such that $|f(t)| \leq Me^{ct}$ for all $t > t_0$. Suppose the integral

$$\phi(\alpha) = \int_a^{\infty} f(x,\alpha)dx,$$

converges for $\alpha_1 \leq \alpha \leq \alpha_2$, or briefly $[\alpha_1, \alpha_2]$, then it is said to be *uniformly convergent* in $[\alpha_1, \alpha_2]$ if for each $\epsilon > 0$, we can find a number N depending on ϵ but not on α, such that

$\left|\phi(\alpha) - \int_a^{\infty} f(x,\alpha)dx\right| < \epsilon$ for all $x > N$ and all α in $[\alpha_1, \alpha_2]$.

A special test for uniform convergence called the *Weirstrass-Majorant test* states that if we can find a function $M(x) \geq 0$ such that: (a) $|f(x,\alpha)| \leq M(x)$ for $\alpha_1 \leq \alpha \leq \alpha_2$, $x > \alpha$ and (b) $\int_a^\infty M(x)dx$ converges, then $\int_a^\infty f(x,\alpha)dx$ is *uniformly* and *absolutely convergent* in $\alpha_1 \leq \alpha \leq \alpha_2$. A theorem on uniformly convergent integrals states that if $f(x,\alpha)$ is continuous for $x \geq a$ and $\alpha_1 \leq \alpha \leq \alpha_2$, and if $\int_a^\infty f(x,\alpha)dx$ is uniformly convergent for $\alpha_1 \leq \alpha \leq \alpha_2$, then $\phi(\alpha) = \int_a^\infty f(x,\alpha)dx$ is continuous in $\alpha_1 \leq \alpha \leq \alpha_2$. Moreover, if α_0 is any point of $\alpha_1 \leq \alpha \leq \alpha_2$, one can write

$$\lim_{\alpha\to\alpha_0} \int_a^\infty f(x,\alpha)dx = \int_a^\infty \lim_{\alpha\to\alpha_0} f(x,\alpha)dx.$$

Another theorem asserts that one can integrate $\phi(\alpha)$ with respect to α from α_1 to α_2 to get

$$\int_{\alpha_1}^{\alpha_2} \phi(\alpha)d\alpha = \int_{\alpha_1}^{\alpha_2} \left\{\int_a^\infty f(x,\alpha)dx\right\} d\alpha = \int_a^\infty \left\{\int_{\alpha_1}^{\alpha_2} f(x,\alpha)d\alpha\right\} dx.$$

These results show that for uniformly convergent integrals, one can interchange the limits and integration, and also the order of integration.

Another result that will be used in the alternative proof for Frullani's theorem is the Fubini's theorem. This theorem provides a means to evaluate a double integral using an iterated integral in which the order of integration can be changed (Love, 1969; Aksoy & Martelli, 2002), provided that that the double integral yields a finite value.The Laplace transform of a continuous function will also play a role in the alternative proof. By definition, the Laplace transform of a continuous function $f(x)$ defined for $x \geq 0$, denoted by $\mathcal{L}\{f(x)\}$, is a continuous function $F(s)$ given by the improper integral (Schiff, 1999; Widder, 2015)

$$\mathcal{L}\{f(x)\} = \int_a^\infty f(x)e^{-sx}dx = F(s).$$

The Laplace transform is an integral transform, which is next to the Fourier transform in its utility in solving physical problems encountered in science and engineering.

The foregoing discussion justifies the following lemmas which are necessary for arriving at the alternative proof of Frullani's theorem or in deriving the Frullani integral formula.

Lemma 3.1.1 If the function f is of exponential order, then there exist $\gamma_1, \gamma_2 \in \mathbb{R}$ such that

$$g(s) = \int_0^\infty [f(ax) - f(bx)]e^{-sx} dx$$

converges absolutely and uniformly for all $s > \max\{\gamma_1, \gamma_2\}$.

Proof: Since f is of exponential order, there exist constants $\gamma_1, \gamma_2 \in \mathbb{R}$ and $M_1, M_2 > 0$ such that when the triangle inequality is applied, the following inequality holds:

$$|f(ax) - f(bx)|e^{-sx} \leq M_1 e^{-(s-\gamma_1)x} + M_2 e^{-(s-\gamma_2)x}$$

for all $x > 0$. By putting $M = \max\{M_1, M_2\}$ and $\gamma = \max\{\gamma_1, \gamma_2\}$, we see that for all $x > 0$ and $s > \gamma$,

$$|f(ax) - f(bx)|e^{-sx} \leq M e^{-(s-\gamma)x} \leq M e^{-x}. \quad (3.1)$$

The integral of the right-hand function in (3.1) converges for all $x > 0$. Thus by Weirstrass-Majorant Test, the integral in the lemma above converges absolutely and uniformly for all $s > \gamma$. Q.E.D

Lemma 3.1.2 If the function f is of exponential order, then $g(s)$ is Riemann integrable on $(\gamma, +\infty)$ where γ is the constant obtained in the proof of lemma 3.1.1.

Proof: Since $g(s)$ as an improper integral with parameter s is uniformly convergent for all $s > \gamma$ (by lemma 3.1.1), then by interchanging the integrals we see that

$$\int_\gamma^\infty g(s)\, ds = \int_0^\infty \int_\gamma^\infty [f(ax) - f(bx)]e^{-sx} ds\, dx$$
$$= \frac{1}{\gamma} \int_0^\infty [f(ax) - f(bx)]e^{-\gamma x}.$$

f being of exponential order implies the right-hand integral above is convergent. Q.E.D

Lemma 3.1.3 If f is of exponential order, then

$$\int_0^\infty \frac{f(ax) - f(bx)}{x} e^{-\alpha x} dx = \int_\alpha^\infty \left[\frac{1}{a}\hat{f}\left(\frac{s}{a}\right) - \frac{1}{b}\hat{f}\left(\frac{s}{b}\right)\right] ds,$$

where $\hat{f}(s) = \int_0^\infty f(x)e^{-sx} dx$ is the Laplace transform of $f(x)$.

Proof: Let $w = ax$, then $dw = adx$ and

$$\int_0^\infty f(ax)e^{-sx}dx = \frac{1}{a}\int_0^\infty f(w)e^{-\frac{s}{a}w}dw = \frac{1}{a}\hat{f}\left(\frac{s}{a}\right). \qquad (3.2)$$

Similarly,

$$\int_0^\infty f(bx)e^{-sx}dx = \frac{1}{b}\hat{f}\left(\frac{s}{b}\right). \qquad (3.3)$$

Subtracting Eq. (3.3) from Eq. (3.2) we have,

$$\int_0^\infty [f(ax) - f(bx)]e^{-sx}dx = \frac{1}{a}\hat{f}\left(\frac{s}{a}\right) - \frac{1}{b}\hat{f}\left(\frac{s}{b}\right).$$

Now, by Lemma 3.1.2, the improper integral $\int_0^\infty [f(ax) - f(bx)]e^{-sx}dx$ is integrable on (γ, ∞) and

$$\int_\alpha^\infty \left\{\int_0^\infty [f(ax) - f(bx)]e^{-sx}dx\right\} ds = \int_\alpha^\infty \left[\frac{1}{a}\hat{f}\left(\frac{s}{a}\right) - \frac{1}{b}\hat{f}\left(\frac{s}{b}\right)\right] ds.$$

Because the foregoing double integral yields a finite value when the integrand is replaced by its absolute value, then by Fubini's theorem, one can switch the order of integration as follows:

$$\begin{aligned}
\int_\alpha^\infty \left\{\int_0^\infty [f(ax) - f(bx)]e^{-sx}dx\right\} ds &= \lim_{c\to\infty}\int_\alpha^c \left\{\int_0^\infty [f(ax) - f(bx)]e^{-sx}dx\right\} ds \\
&= \lim_{c\to\infty}\int_0^\infty \left\{\int_\alpha^c [f(ax) - f(bx)]e^{-sx}ds\right\} dx \\
&= \lim_{c\to\infty}\int_0^\infty \left\{[f(ax) - f(bx)]\int_\alpha^c e^{-sx}ds\right\} dx \\
&= \lim_{c\to\infty}\int_0^\infty \left\{\int_\alpha^c [f(ax) - f(bx)]\frac{-e^{-sx}}{x}\Big|_\alpha^c\right\} dx \\
&= \int_0^\infty \frac{f(ax)-f(bx)}{x}e^{-\alpha x}dx.
\end{aligned}$$

Thus,

$$\int_0^\infty \frac{f(ax)-f(bx)}{x}e^{-\alpha x}dx = \int_\alpha^\infty \left[\frac{1}{a}\hat{f}\left(\frac{s}{a}\right) - \frac{1}{b}\hat{f}\left(\frac{s}{b}\right)\right] ds. \text{ Q.E.D}$$

Lemma 3.1.4 If the function f is of exponential order and

$$\int_0^\infty \frac{f(ax)-f(bx)}{x}dx$$

is convergent, then

$$\int_0^\infty \frac{f(ax)-f(bx)}{x}dx = \int_0^\infty \left[\frac{1}{a}\hat{f}\left(\frac{s}{a}\right) - \frac{1}{b}\hat{f}\left(\frac{s}{b}\right)\right] ds.$$

Proof: From Lemma 3.1.3, we have

$$\int_0^\infty \frac{f(ax)-f(bx)}{x}e^{-\alpha x}dx = \int_\alpha^\infty \left[\frac{1}{a}\hat{f}\left(\frac{s}{a}\right) - \frac{1}{b}\hat{f}\left(\frac{s}{b}\right)\right] ds,$$

where $\alpha \in (0, \infty)$.

As the proof of lemma 3.1.1, the improper integral $\int_0^\infty \frac{f(ax)-f(bx)}{x}e^{-\alpha x}dx$ is uniformly convergent for all $\alpha \in (0, \infty)$. Consequently, we can interchange the limit and the integral, that is,

$$\lim_{\alpha\to 0^+}\int_0^\infty \frac{f(ax)-f(bx)}{x}e^{-\alpha x}dx = \int_0^\infty \lim_{\alpha\to 0^+}\frac{f(ax)-f(bx)}{x}e^{-\alpha x}dx$$
$$= \int_0^\infty \frac{f(ax)-f(bx)}{x}dx. \quad (3.4)$$

Moreover,

$$\lim_{\alpha\to 0^+}\int_0^\infty \frac{f(ax)-f(bx)}{x}e^{-\alpha x}dx = \lim_{\alpha\to 0^+}\int_\alpha^\infty \left[\frac{1}{a}\hat{f}\left(\frac{s}{a}\right) - \frac{1}{b}\hat{f}\left(\frac{s}{b}\right)\right] ds$$
$$= \int_0^\infty \left[\frac{1}{a}\hat{f}\left(\frac{s}{a}\right) - \frac{1}{b}\hat{f}\left(\frac{s}{b}\right)\right] ds. \quad (3.5)$$

Comparing equations (3.4) and (3.5) we get

$$\int_0^\infty \frac{f(ax)-f(bx)}{x}dx = \int_0^\infty \left[\frac{1}{a}\hat{f}\left(\frac{s}{a}\right) - \frac{1}{b}\hat{f}\left(\frac{s}{b}\right)\right] ds.$$

3.2 Alternative proof of Frullani's theorem

The alternative proof that we shall present requires the additional condition that $\int_0^\infty f'(x)dx$ is convergent in order for the solution of the Frullani integral to exist.

Theorem 3.2.1 (Frullani integral) If f is of exponential order and $\int_0^\infty f'(x)dx$ is convergent, then

$$\int_0^\infty \frac{f(ax)-f(bx)}{x}dx = \ln\left(\frac{a}{b}\right)[f(\infty)-f(0)],$$

where $f(\infty) = \lim_{x\to\infty} f(x)$ and $f(0) = \lim_{x\to 0^+} f(x)$.

Proof: By Lemma 3.1.4, we have

$$\int_0^\infty \frac{f(ax)-f(bx)}{x}dx = \int_0^\infty \left[\frac{1}{a}\hat{f}\left(\frac{s}{a}\right) - \frac{1}{b}\hat{f}\left(\frac{s}{b}\right)\right]ds,$$

where $\hat{f}(s)$ is the Laplace Transform of f. Now by definition of Laplace transform, we have

$$\left[\frac{1}{a}\hat{f}\left(\frac{s}{a}\right) - \frac{1}{b}\hat{f}\left(\frac{s}{b}\right)\right] = \int_0^\infty \left[\frac{1}{a}e^{-\frac{s}{a}x} - \frac{1}{b}e^{-\frac{s}{b}x}\right]f(x)dx.$$

Integrating both sides of the equation above with respect to s, we have

$$\int_0^\infty \left[\frac{1}{a}\hat{f}\left(\frac{s}{a}\right) - \frac{1}{b}\hat{f}\left(\frac{s}{b}\right)\right]ds = \int_0^\infty \left\{\int_0^\infty \left[\frac{1}{a}e^{-\frac{s}{a}x} - \frac{1}{b}e^{-\frac{s}{b}x}\right]f(x)dx\right\}ds.$$

Now, by integration by parts, with $u = f(x)$ and $dv = \left[\frac{1}{a}e^{-\frac{s}{a}x} - \frac{1}{b}e^{-\frac{s}{b}x}\right]dx$, we see that

$$\int_0^\infty \left[\frac{f(ax)-f(bx)}{x}\right]dx = \int_0^\infty \int_0^\infty \frac{e^{-\frac{s}{a}x} - e^{-\frac{s}{b}x}}{s} f'(x)dxds.$$

Since the conditions of Fubini's theorem are satisfied, that is, the double integral gives a finite answer when the integrand is replaced by its absolute value, then the order of integration on the right side expression can be changed (Love, 1969; Aksoy & Martelli, 2002) so that,

$$\int_0^\infty \left[\int_0^\infty \frac{e^{-\frac{s}{a}x} - e^{-\frac{s}{b}x}}{s} f'(x)dx\right] ds = \int_0^\infty \left[\int_0^\infty \frac{e^{-\frac{s}{a}x} - e^{-\frac{s}{b}x}}{s} f'(x)ds\right] dx$$

$$= \int_0^\infty f'(x) \left[\int_0^\infty \frac{e^{-\frac{s}{a}x} - e^{-\frac{s}{b}x}}{s} ds\right] dx$$

$$= \int_0^\infty f'(x) \ln\left(\frac{a}{b}\right) dx.$$

Thus,

$$\int_0^\infty \left[\frac{f(ax)-f(bx)}{x}\right] dx = \int_0^\infty f'(x) \ln\left(\frac{a}{b}\right) dx = \ln\frac{a}{b}\int_0^\infty f'(x)dx.$$

$$= \ln\frac{a}{b}(f(\infty) - f(0))$$

where $f(\infty) = \lim_{x\to\infty} f(x)$ and $f(0) = \lim_{x\to 0^+} f(x)$. Q.E.D

3.3 A novel method for evaluating Frullani's integral

As a consequence of Lemma 3.1.3, a new method for solving certain special improper integrals of Frullani type emerged, as we have shown that

$$\int_0^\infty \frac{f(ax)-f(bx)}{x} dx = \int_0^\infty \left[\frac{1}{a}\hat{f}\left(\frac{s}{a}\right) - \frac{1}{b}\hat{f}\left(\frac{s}{b}\right)\right] ds, \tag{3.6}$$

where

. $$\hat{f}(s) = \int_0^\infty f(x)e^{-sx}dx.$$

This time we will be applying equation (3.6) to evaluate some improper integrals of frullani-type.

Illustration 3.3.1 Evaluate $\int_0^\infty \frac{e^{-ax}-e^{-bx}}{x}dx$;

Solution: Let $f(x) = e^{-x}$. Let us note that,

$$\hat{f}(s) = \int_0^\infty (e^{-x})e^{-sx}dx = \int_0^\infty e^{-(1+s)x}dx = \frac{1}{1+s}.$$

Now, applying Eq. (3.6) we have

$$\int_0^{\infty} \frac{e^{-ax} - e^{-bx}}{x} dx$$
$$= \int_0^{\infty} \left[\frac{1}{a} \left(\frac{1}{1 + \frac{s}{a}} \right) - \frac{1}{b} \left(\frac{1}{1 + \frac{s}{b}} \right) \right] ds$$
$$= \lim_{x \to \infty} \int_0^{x} \left[\frac{1}{s + a} - \frac{1}{s + b} \right] ds$$
$$= \lim_{x \to \infty} \left\{ (\ln(s + a) - \ln(s + b)) \Big|_0^x \right\}$$
$$= \lim_{x \to \infty} \left\{ \ln \left(\frac{s + a}{s + b} \right) \Big|_0^x \right\}$$
$$= \lim_{x \to \infty} \ln \left(\frac{x + a}{x + b} \right) - \ln \left(\frac{a}{b} \right).$$
$$= -\ln \left(\frac{a}{b} \right)$$
$$= \ln \left(\frac{b}{a} \right).$$

Illustration 3.3.2 Evaluate $\int_0^{\infty} \frac{\cos(ax) - \cos(bx)}{x} dx$.

Solution: Let $f(x) = \cos x$. let us note that,

$$\hat{f}(s) = \int_0^{\infty} (\cos x) e^{-sx} dx = \frac{s}{s^2 + 1}.$$

By applying equation (3.6) again,

$$\int_0^{\infty} \frac{\cos(ax) - \cos(bx)}{x} dx = \int_0^{\infty} \left[\frac{1}{a} \left(\frac{\frac{s}{a}}{\left(\frac{s}{a}\right)^2 + 1} \right) - \frac{1}{b} \left(\frac{\frac{s}{b}}{\left(\frac{s}{b}\right)^2 + 1} \right) \right] ds$$
$$= \lim_{x \to \infty} \int_0^{\infty} \left[\frac{s}{a^2 + s^2} - \frac{s}{b^2 + s^2} \right] ds$$
$$= \lim_{x \to \infty} \left\{ \left(\frac{1}{2} \ln(s^2 + a^2) - \frac{1}{2} \ln(s^2 + b^2) \right) \Big|_0^x \right\}$$

$$= \lim_{x\to\infty}\left\{\frac{1}{2}\ln\left(\frac{x^2+a^2}{x^2+b^2}\right)-\frac{1}{2}\ln\left(\frac{a^2}{b^2}\right)\right\}.$$
$$= -\frac{1}{2}\ln\left(\frac{a^2}{b^2}\right)$$
$$= \ln\left(\frac{b}{a}\right).$$

An interesting consequence of the Alternative Proof of Theorem 3.2.1 is that we were able to relate the Frullani integral to a class of improper double integrals over $[0,\infty)\times[0,\infty)$. Specifically, we have shown that

. $$\int_0^\infty \frac{f(ax)-f(bx)}{x}dx = \int_0^\infty\int_0^\infty \frac{e^{-\frac{t}{a}x}-e^{-\frac{t}{b}x}}{t} f'(x)dxdt. \tag{3.7}$$

The following are illustrative examples aimed to show how this integral equation can be used for evaluating certain class of improper double integrals by relating it to the Frullani integral (Jung, 2012).

Illustration 3.3.3

Show that $\int_0^\infty\int_0^\infty \frac{e^{-\frac{t}{b}x}-e^{-\frac{t}{a}x}}{t} e^{-x}dxdt = \ln\left(\frac{b}{a}\right)$.

Proof: Let $f(x) = e^{-x}$. Then $f'(x) = -e^{-x}$. Then by illustration 3.1.1,

$$\int_0^\infty \frac{e^{-ax}-e^{-bx}}{x}dx = \ln\left(\frac{b}{a}\right).$$

Thus by equation (3.7),

$$\int_0^\infty\int_0^\infty \frac{e^{-\frac{t}{b}x}-e^{-\frac{t}{a}x}}{t} e^{-x}dxdt = \ln\left(\frac{b}{a}\right).$$

Similarly when $f(x) = e^{-x^2}$, we obtain $\int_0^\infty\int_0^\infty \frac{e^{-\frac{t}{b}x}-e^{-\frac{t}{a}x}}{t} xe^{-x^2}dxdt = \frac{1}{2}\ln\left(\frac{b}{a}\right)$.

4. CONCLUSIONS

This work presented an alternative proof for the Frullani's integral formula. Moreover, the manner or the approach used in proving the Frullani integral formula yielded a novel approach for evaluating improper integrals of Frullani type and certain improper double integrals. Several illustrations were provided to demonstrate how to apply the novel method.

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