ON THE HAMILTONICITY OF PRODUCT GRAPH $G \square S_m$, FOR A GRAPH G OF ORDER n, AND STAR GRAPH S_m, $n \geq m$

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Abstract: Given two graphs G and H, the cartesian product, $G \square H$ is the graph whose vertex set is $V(G) \times V(H)$ and the set $\{(u1, v1), (u2, v2)\}$ is an edge if and only if exactly one of the following is true.

(i) $u_1 = u_2$ *and* $\{v_1, v_2\}$ is an edge in H.
(ii) $v_1 = v_2$ *and* $\{u_1, u_2\}$ is an edge in G.

A *star graph* S_m, also known as a *complete bipartite* graph $K_{1,m}$, is a graph whose vertex set consists of two disjoint sets $V_1 = \{c\}$ and $V_2 = \{v_1, v_2, \ldots, v_m\}$, known as partites, such that no two vertices in V_2 are adjacent but all of them are adjacent to c. A *hamiltonian graph* is a graph that contains a cycle containing all its vertices. Clearly, S_m is not hamiltonian for all $m \geq 1$.

In this paper the following shall be proven:

Let *G* be a hamiltonian graph, Cn be a cycle graph and Kn be a complete graph, all of orders *n*, and S_m be a star graph, $m \geq 1$, then

1. $C \square Sm$ is hamiltonian if and only if $n \geq m$, $n \geq 3$
2. $Kn \square Sm$ is hamiltonian if and only if $n \geq m$, $n \geq'' 2$
3. $G \square Sm$ is hamiltonian if and only if $n \geq m$.

***Keywords**: Hamiltonian, Cartesian, bipartite, star graph*

1. Some Preliminaries

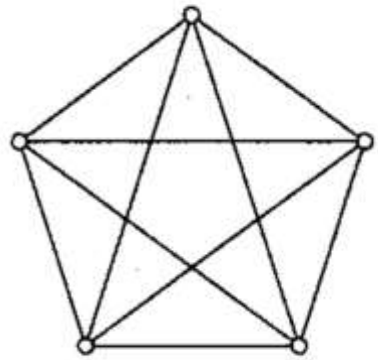
For better understanding of the paper, some terms will be defined.

A *graph* Γ consists of an ordered pair $(V(\Gamma), E(\Gamma))$ where $V(\Gamma)$ is a non-empty set and $E(\Gamma)$ is either a set of two element subsets of $V(\Gamma)$ or is empty. The elements of $V(\Gamma)$ are called vertices and the elements of $E(\Gamma)$ are called edges. If $\{u, v\}$ is an edge of Γ then we say that u and v are adjacent to one another.

The number of vertices of a graph is known as its order and the number of edges is called size. We usually symbolize by $p(\Gamma)$ *and* $q(\Gamma)$ respectively. The degree of a vertex u, $deg(u)$ is the number of vertices in Γ adjacent to u.

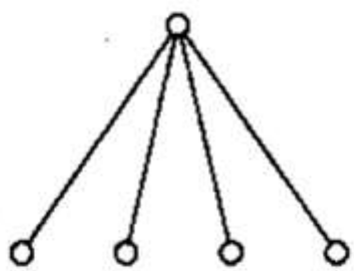
The complete graph K_n is a graph whose order is n and every distinct vertices are adjacent to one another. Thus, the degree of every vertex in K_n is $n-1$. The complete bipartite $K_{r,m}$ is a graph whose vertex set $V(K_{r,m})$ is a union of two disjoint sets V_1 *and* V_2, known as the partites such that if two vertices are in the same partite then, they are not adjacent. Furthermore, every vertex in one partite is adjacent to every vertex of the other partite. $K_{1,m}$ is also known as star graph and is denoted by S_m. Let $V(S_m) = V_1 \cup V_2$, where $V_1 = \{c\}$ and $V_2 = \{v_1, v_2 \ldots, v_m\}$ be the distinct partites of S_m, then we shall call c as the center of S_m.

A graph may be illustrated as follows: small circles or dots may represent the vertices, and the edges may be represented by lines or curves joining vertices which are adjacent to one another.



K5

Figure 1: Complete graph of order 5



S4

Figure 2. Complete bipartite graph $K_{1,4}$ *or* S_4

2. Cartesian Product of Graphs

Given graphs G and H, a new graph may be formed known as the Cartesian product of G and H written as $G \square H$. *If* $\Gamma = G \square H$, *then* $V(\Gamma) = V(G) \times V(H)$ and the set $\{(u_1, v_1), (u_2, v_2)\}$ is an edge if and only if exactly one of the following is true:

1. $u_1 = u_2$ and $\{v_1, v_2\}$ is an edge in H, or
2. $v_1 = v_2$ *and* $\{u_1, u_2\}$ *is an edge in* G.

Intuitively, the cartesian product $G\square Sm$ is a graph formed by "replacing" each vertex of S_m with G and edges are formed according to definition.

Example 2.1. Consider $\Gamma = K_1 \square S_2$.

Let $V(K_2) = \{u_1, u_2\}$ *and* $V(S_2) = \{v_1, v_2\} \cup \{c\}$. Then,

$V(\Gamma) = \{(u_1, v_1)(u_1, v_2)(u_1, c)(u_2, v_1)(u_2, v_2)(u_2, c)\}$

$E(\Gamma) = \{\{(u_1, v_1), (u_1, c)\}, \{(u_1, v_2)(u_1, c)\}, \{(u_2, v_1), (u_2, c)\}\{(u_2, v_2)(u_2, c)\}, \{(u_1, v_1), (u_2, v_1)\}, \{(u_1, v_2), (u_2, v_2)\}, \{(u_1, c)(u_2, c)\}\}$

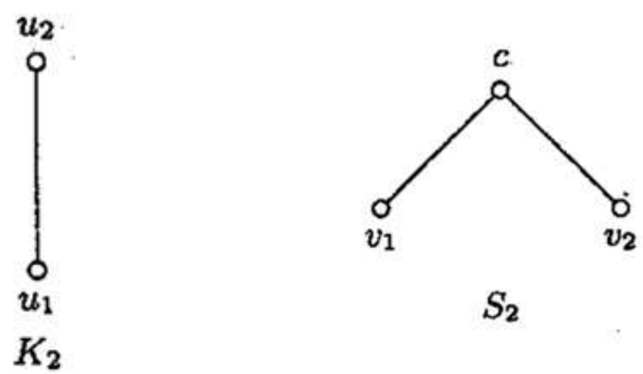


Figure 3. Complete graph K_2 and star graph S_2

Figure 4 below is the Cartesian product of $K2\square S2$. One can see that each vertex of S_2 was replaced by K_2 and corresponding adjacency among vertices were made.

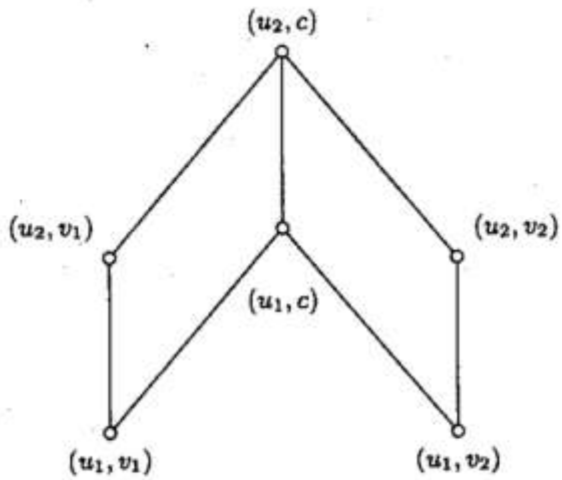


Figure 4: Cartesian product of K_2 and S_2

3. Hamiltonian Graph

A path P_k of a graph is sequence of adjacent vertices $u_2, u_2, . . . u_k$ such that no one vertex is repeated. A closed path or cycle is a sequence of adjacent vertices $u_1, u_2, . . . u_k, u_{k+1}$ such that $u_1 = u_{k+1}$ and no other vertex is repeated in the sequence. A *cycle graph* or *n- cycle* C_n is a graph of order n and whose vertices form a cycle.

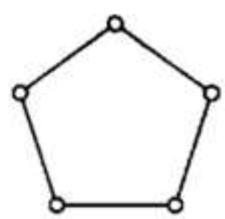


Figure 5: Graph of the cycle graph C_5

A graph H is said to be hamiltonian if we can find a cycle in H that contains all its vertices. This cycle is known as a hamiltonian cycle or a spanning cycle. Notice that a complete graph K_n, $n \geq 3$ is Hamiltonian while a star graph is not.

It is a known fact in graph theory that the cartesian product of hamiltonian graphs is again hamiltonian. But what about the cartesian product of graphs of which one is not hamiltonian? This paper will prove the following.

Let G be a hamiltonian graph , C_n be a cycle graph and K_n be a complete graph, all of orders n, and S_m be a star graph, $m \geq 1$, then

1. $C_n \square S_m$ is hamiltonian if and only if $n \geq m$, $n \geq 3$

2. $K_n \square S_m$ is hamiltonian if and only if $n \geq m$, $n \geq 2$

3. $G \square S_m$ is hamiltonian if and only if $n \geq m$.

Proposition 3.1. The cartesian product $C_n \square S_m$, $m \geq 1$ is hamiltonian if and only if $n \geq m$ *for* $n \geq 3$.

Proof.

Let $\Gamma = C_n \square S_m$. Without loss of generality, assume $V(C_n) = \{u_1, u_2, \ldots, u_n\}$ and that $u_1, u_2, \ldots, u_n, u_1$ is its cycle. Also, let $V(S_m) = \{c\} \cup \{v_1, v_2, \ldots, v_m\}$, where c is the center of S_m. Then:

$$V(\Gamma) = \bigcup_{i=1}^{n} \{(u_i, v_j) | j = 1, 2, \ldots, m\} \cup \{(u_i, c) | i = 1, 2, \ldots, n\}$$

$$E(\Gamma) = \bigcup_{i=1}^{n} \{\{(u_i, v_j), (u_i, c)\} | j = 1, 2, \ldots, m\} \cup \left[\bigcup_{1}^{n-1} \{\{(u_i, v_j), (u_{i+1}, v_j)\} | j = 1, 2, \ldots, m\} \right]$$
$$\cup \{\{(u_1, v_j), (u_n, v_j)\} | j = 1, 2, \ldots, m\} \cup \left[\bigcup_{i=1}^{n-1} \{\{(u_i, c), (u_{i+1}, c)\}\} \right] \cup \{\{(u_1, c), (u_n, c)\}\}$$

Suppose Γ is hamiltonian, then there exists a cycle C that contains all the vertices of Γ. Consequently, C will contain a path that will pass through all the vertices of Γ. Notice that all paths connecting vertices with second coordinates v_i and v_j (distinct), must contain a vertex (u, c) for some $u \in V(C_n)$. Thus, there exists $n-m$ vertices with second coordinate c that will be left "unvisited", after any path connecting vertices with second coordinates $v_1, v_2, \ldots v_m$ have been constructed. It follows that $n-m \geq 0$ or $n \geq m$. Since C_n is a cycle, then n " 3.

Suppose $n \geq m$, and C_n is a cycle, then $n \geq 3$. Also, m " 1. So, $n - m \geq 0$. Consequently, $n - (m-1)$ 1. Thus, $1 \leq n - (m-1) \leq n$. Consider now the following table.

Partites					c
v_1	u_1	u_2	u_3	$\cdots$	u_n
v_2	u_n	u_1	u_2	$\cdots$	u_{n-1}
v_3	u_{n-1}	u_n	u_1	$\cdots$	u_{n-2}
$\vdots$	$\vdots$	$\ddots$			$\vdots$
v_m	$u_{n-(m-2)}$	$u_{n-(m-3)}$	$u_{n-(m-4)}$	$\cdots$	$u_{n-(m-1)}$
					u_{n-m}
					$\vdots$
					u_1

Table 1: Guide to constructing the Hamiltonian cycle

Using the above table as guide, we now form the following cycle:

$$(u_1, v_1), (u_2, v_1), (u_3, v_1), \ldots, (u_n, v_1)(u_n, c), (u_n, v_2), (u_1, v_2), (u_2, v_3), \ldots,$$
$$(u_{n-1}, v_2), (u_{n-1}, c), (u_{n-1}, v_3), (u_n, v_3), (u_1, v_3), \ldots (u_{n-2}, c), \ldots, (u_m, c), (u_m, v_m),$$
$$(u_{m+1}, v_m), (u_{m+2}, v_m) \ldots, (u_{n-(m-1)}, c), (u_{n-m}, c), \ldots, (u_1, c), (u_1, v_1)$$

Clearly, above is a spanning cycle of Γ and thus, it is hamiltonian.

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We now consider the case of $K_n \square S_m$. For $n = 1, 2, K_n$ does not contain a cycle. We shall prove however that for $n \geq 2$ *and* $n \geq m$, the cartesian product is hamiltonian.

Proposition 3.2.

The product graph $K_n \square S_m$, $m \geq 1$, is hamiltonian if and only if $n \geq m$ and $n \geq 2$.

Proof.

Let $V(Kn) = \{u_1, u_2, . . . , u_n\}$ and $V(S_m) = \{v_1, v_2, . . . , v_m\} \cup \{c\}, c \neq v_k$ for all k.

Note that for all $i \neq j$, u_i is adjacent to u_j and the elements of $S = \{v_1, v_2, . . . , v_m\}$ are non-adjacent vertices but each of the vertex in S is adjacent to c.

Let $\Gamma = K_n \square S_m$ be hamiltonian. If $n = 1$ and $m \geq 1$ then K_1 is simply a single vertex and $K_1 \square S_m \cong S_m$, which is not Hamiltonian. This contradicts the assumption that Γ is hamiltonian. For $\Gamma = K_2 \square S1$, it is simply C_4. Clearly this is Hamiltonian. Thus, $n \geq 2$. Also, since Γ is hamiltonian, then there exists a cycle C that contains all the vertices of Γ. From proof of previous theorem, it follows that $n \geq m$.

Conversely, let $n \geq 2$, and $n \geq m$. If $n = 2$, then $m = 1, 2$. Now, $K_2 \square S1$ is C_4 and thus hamiltonian. $K_2 \square S_2$ is just the graph

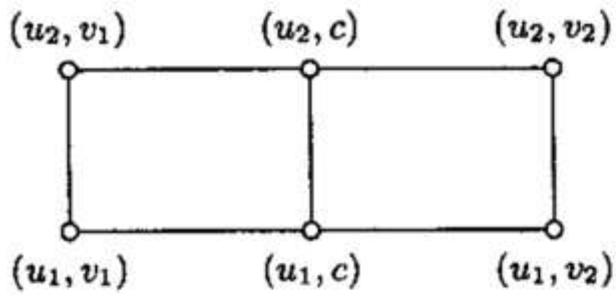

Figure 6: Graph of $K_2 \square S_2$

The cycle $(u_1, v_1), (u_1, c), (u_1, v_2), (u_2, v_2), (u_2, c), (u_2, v_1), (u_1, v_1)$ is the required hamiltonian cycle.

For $n \geq 3$, K_n is hamiltonian and thus contains a spanning cycle. From the proof of previous proposition, there exists a spanning cycle for $K_n \square S_m$. It follows that the graph is hamiltonian. Therefore, $K_n \square S_m$ is hamiltonian whenever $n \geq 2$.

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Example 3.1.

Γ:

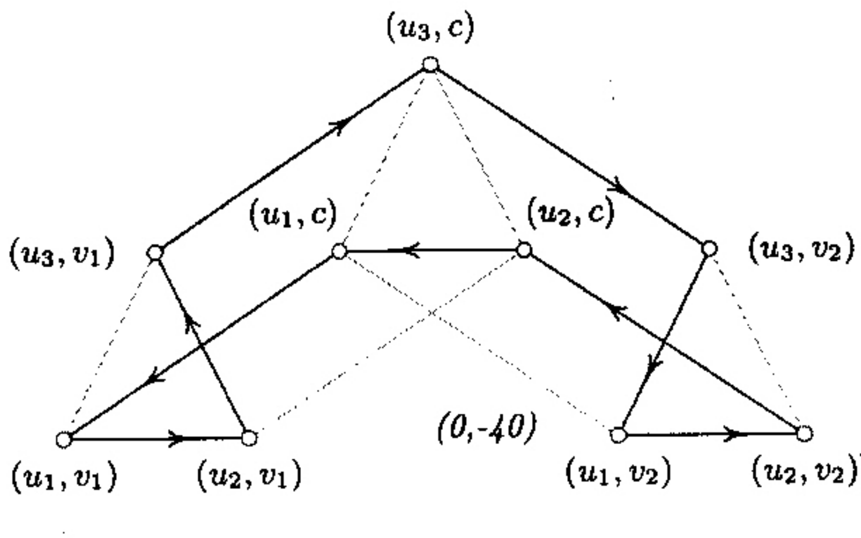

Figure 7. Hamiltonian cycle of $K_2\Box S_2$

Lastly, we prove the more general case for any graph G of order m.

Proposition 3.3. Let G be a hamiltonian graph of order n. Then $G\Box S_m$, $m \geq 1$ is hamiltonian if and only if $n \geq m$ and $n \geq 3$.

Proof.

If G is hamiltonian, then G contains a spanning cycle. Hence, its order n must be greater than or equal to 3. We can now apply Proposition 3.1 and thus theorem is proved.

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The results in this paper are the initial results of the ongoing study on the hamiltonicity of the product graph $G\Box K_{r,m}$ where G is a graph of order n and $K_{r,m}$ is the complete bipartite graph and $r \geq 1$.

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